Mathematics - Brush-up
Problem Set 2

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1 Differentiability

Exercise 1
Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^{\frac{1}{2}} y^{\frac{1}{2}}$. Compute the partial derivatives and evaluate them at $(x, y) = (0, 0)$. Is f differentiable at $(0, 0)$? Is f continuously differentiable?

Exercise 2
Compute the partial derivatives, if they exist, and check the differentiability at $(x, y) = (0, 0)$ of the following functions:

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = |x + y|$

Exercise 3
Find the Jacobian matrix at the point (x, y) of the function $h = gf$ where $f(x, y) = (x^2, \exp^{x+y}, x - y)$ and $g(u, v, z) = (u - v^2, \exp^z)$.

Exercise 4
Find the points where the function $f(x, y) = (x^2 + y^2, xy)$ has a local inverse. Find the Jacobian matrix of the inverse function, at the points that exists.

2 Optimization

Exercise 5
Consider the following Lagrange problem:

$$\begin{aligned} optimize_{(x,y,z)} \quad f(x, y, z) = 3x + 2y + z \\ s.t. \\ x^2 + y^2 + z^2 = 2 \\ x + y + z = 0 \end{aligned}$$

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1. Show that all points are regular.
2. Compute the candidates for local optima of the problem.
3. Determine local maximum and minimum.

Exercise 6
Consider the function $f(x, y) = x + y$ and the feasible set determined by:

$$\begin{cases} y - x^3 = 0 \\ x^4 - y = 0. \end{cases}$$

1. Check that the point $(0, 0)$ is the only in the feasible set that is not regular.
2. Compute the points that satisfy the Kuhn-Tucker conditions and their respective multipliers.
3. Give an argument to decide if the function f has a maximum and/or a minimum over the feasible set and find them in the affirmative case.

Exercise 7
Given the following optimization problem:

$$\begin{aligned} optimize_{(x,y,z)} f(x, y) = \exp^{x^2+y^2} \\ s.t. \\ y \geq x^2 + 1 \\ x \geq 1 \end{aligned}$$

1. Show that it is a convex problem.
2. Check if Slater′s condition is satisfied.
3. Show that $(1, 2)$ is the global minimum of the problem and that there is no global maximum.

3 Integration

Exercise 8
Let $f(x) = x$. Compute $U(f, P)$ and $L(f, P)$ for a generic n-partition P. Use these formulas and the Riemann integrability criterion to prove that f is Riemann integrable on $[0, 1]$ and to prove that $\int_0^1 f(x)dx = \frac{1}{2}$

Exercise 9
Let $a \leq c < d \leq b$ and let $f : [a, b] \to \mathbb{R}$ be a function defined by:

$$f(x) = \begin{cases} 0 & \text{if} \quad a \leq x \leq c \\ 1 & \text{if} \quad c < x < d \\ 0 & \text{if} \quad d \leq x \leq b \end{cases}$$

Prove that f is Riemann integrable on $[a, b]$ and that $\int_a^b f(x)dx = (d - c)$.

Exercise 10

Consider the cumulative distribution function of a r.v. X, defined by:

$$Prob(X \leq x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ p & \text{if} \quad 1 \leq x \leq 3 \\ 1 & \text{if} \quad x \geq 3 \end{cases}$$

If we let α be equal to $Prob(X \leq x)$, then α will be an increasing function, suited for the Riemann-Stieltjes integrals. Then we can write:

$$\alpha(x) = Prob(X \leq x) = pJ_1(x) + (1-p)J_3(x)$$

where

$$J_c(x) = \begin{cases} 0 & \text{if} \quad x < c \\ 1 & \text{if} \quad x \geq c \end{cases}$$

Notice that in this case, if f is continuous then is Riemann-Stieltjes integrable, and $\int_a^b f d\alpha(x) = pf(c)$, for $c \in (a, b]$. Let $f(x) = x$. For $a > 1$ and $b < 3$, calculate $\int_a^b f d\alpha(x)$.